On the curvature of some free boundaries in higher dimensions

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Abstract. It is known that any subharmonic quadrature domain in two dimensions satisfies a natural inner ball condition, in other words there is a specific upper bound on the curvature of the boundary. This result directly applies to free boundaries appearing in obstacle type problems and in Hele-Shaw flow. In the present paper we make partial progress on the corresponding question in higher dimensions. Specifically, we prove the equivalence between several different ways to formulate the inner ball condition, and we compute the Brouwer degree for a geometrically important mapping related to the Schwarz potential of the boundary. The latter gives in particular a new proof in the two dimensional case.

Keywords. quadrature domain, inner ball condition, Schwarz potential, Brouwer degree.
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1. Introduction

In the present paper we study the curvature of a free boundary which comes up in some obstacle type problems, more specifically Laplacian growth (Hele-Shaw flow moving boundary problem), quadrature domains and partial balayage. The final aim of the investigations is to show that the free boundary in question satisfies a certain inner ball condition (a specific upper bound on the curvature). This goal has previously been achieved in the case of two dimensions (see [7] and [8]; compare also [14]). Here we shall give some partial results (but no complete solution) in higher dimensions, and in passing also obtain a new proof for the two dimensional case.

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The geometric property we aim at proving can most easily be stated in terms of quadrature domains for subharmonic functions [10]. Let μ a positive Borel measure with compact support in $\mathbb{R}^n$. A bounded open set $\Omega \subset \mathbb{R}^n$ is called a *quadrature open set* for subharmonic functions with respect to μ if $\mu(\mathbb{R}^n \setminus \Omega) = 0$ and

$$\int_\Omega h\, d\mu \le \int_\Omega h\, dm$$

for all integrable (with respect to Lebesgue measure dm) subharmonic functions h in Ω. A quadrature open set which is connected is a *quadrature domain.* There is a natural process of balayage of measures to a prescribed density (partial balayage) by which Ω can be constructed from μ when it exists, see e.g. [10], [6], [5]. This balayage process is also equivalent to solving a certain obstacle problem [11].

In terms of the difference $u = U^\mu - U^\Omega$ between the Newtonian potentials of μ and Ω, the latter considered as a body of density one, the quadrature property spells out to

$$\begin{cases} u \ge 0 & \text{in } \mathbb{R}^n, \\ u = 0 & \text{outside } \Omega, \\ \Delta u = \chi_\Omega - \mu & \text{in } \mathbb{R}^n. \end{cases}$$

The function u appearing here is sometimes called the modified Schwarz potential of $\partial\Omega$, see [15] for example. What has been proved in two dimensions, and what we like to extend to higher dimension, is that Ω in the above situation can be written as the union of open balls centered in the closed convex hull K of $\operatorname{supp}\mu$:

$$\Omega = \cup_{x\in K} B(x, r(x)). \tag{1.1}$$

Here $r(x) \ge 0$ denotes the radius of the ball at x, allowing the possibility $r(x) = 0$, i.e., that the ball is empty. We refer to (1.1) as Ω satisfying the inner ball property with respect to K. For further discussion and motivations in the present context, see [7], [8].

The inner ball property concerns the geometry of $\partial\Omega$ outside any closed half-space H containing $\operatorname{supp}\mu$. By a certain "localization" procedure the part $\Omega \setminus H$ of Ω which is outside H can be shown to be identical with a quadrature open set for some positive measure with support on ∂H, see [6], [13]. For this reason it is enough to prove (1.1) in the case that μ has support in a hyperplane, and by a further localization one may even assume μ to have a continuous density on it. For convenience we shall take the hyperplane in question to be $\{x \in \mathbb{R}^n : x_n = 0\}$. It is known (see [10], [6]) that given any positive measure with compact support in this hyperplane, which we identify with $\mathbb{R}^{n-1}$, there is a uniquely determined quadrature open set, which moreover is symmetric about the hyperplane and is convex in the x_n-direction (i.e., the intersection with any straight line perpendicular to the hyperplane is connected). Thus the quadrature open set can be described in terms of a graph of a function g defined in an open subset D of the hyperplane. It is known from the regularity theory of free boundaries (see [4], [1], and in the present context [6]) that this function g is real analytic. It will be enough to consider the

case that D is connected, because in the disconnected case the discussions will apply to each component separately.

In the paper we shall therefore study the geometry of quadrature domains for a measure with support in the hyperplane $\mathbb{R}^{n-1}$, the corresponding modified Schwarz potential u, and various differential geometric objects derived from it. The paper consists of two main parts. The first part starts with some differential geometric preliminaries (Section 2) and ends up with a proof of equivalence of several different formulations of the inner ball condition (Section 3). This part is analogous to a corresponding part in [7], but new difficulties appear in the higher dimensional case. In the second part of the paper, Section 4, we develop tools for studying the geometry by means of vector fields and differential forms defined in terms of u.

To briefly explain, in terms the two dimensional situation, what we do in the second part of the paper, let Ω^+ denote the part of the quadrature domain which lies in the upper half space (half plane) and let $S(z)$ be the Schwarz function (see [2], [15]) of $(\partial\Omega)^+$. This is in our case given by $S(z) = \bar{z} - 4\frac{\partial u}{\partial z}$, it is analytic in Ω^+, and equals $\bar{z}$ on $(\partial\Omega)^+$. What we study in the second part of the paper is the higher dimensional counterpart of the mapping $\sigma : \Omega^+ \to \mathbb{C}$ defined by

$$\sigma(z) = \frac{1}{2} z S(z) = \frac{1}{2} r^2 - r\frac{\partial u}{\partial r} + \mathrm{i}\frac{\partial u}{\partial \varphi},$$

the last member referring to polar coordinates. We also study in higher dimensions that curve γ which in two dimensions is defined by $\frac{\partial u}{\partial \varphi} = 0$. In the two dimensional case a proof of the inner ball property can be based on the topological property of γ that it separates the two domains $\frac{\partial u}{\partial \varphi} < 0$ and $\frac{\partial u}{\partial \varphi} > 0$ from each other, see [8]. There seems to be no direct counterpart of this kind of proof in higher dimensions.

A slightly different proof in two dimensions uses the argument principle for σ, see Corollary 4.9 in the present paper. We have not been able to generalize this proof either to higher dimensions, even though we think that such a proof may not be completely out of reach. At least we have computed the relevant mapping degree of σ in higher dimensions, and this result, Theorem 4.8, may be considered to be the main result in the second part of the paper. Another avenue which we have pursued to some extent is the investigation of the general behaviour of the curve γ. This curve starts out at the origin and reaches $(\partial\Omega)^+$ only at points where the largest inner ball centered at the origin touches $(\partial\Omega)^+$, see Proposition 4.10. If we could establish for example that γ were a smooth curve (no branchings) a proof of the inner ball property would not be far away.

Thus we hope that the partial results we obtain in this paper will turn out to be useful in forthcoming attempts to prove the inner ball property for quadrature domains, explicitly formulated in Conjecture 4.1.

2. Differential geometric preliminaries

2.1. Notations

In this section we decompose the coordinates in $\mathbb{R}^n$ typically as (u, v) where $u = (u_1, ..., u_{n-1}) \in \mathbb{R}^{n-1}$, $v \in \mathbb{R}$. We denote by $(\mathbb{R}^n)^+ = \{(u, v) \in \mathbb{R}^n : v > 0\}$ the upper half space. Let g be a positive, twice continuously differentiable function defined in a bounded domain $D \subset \mathbb{R}^{n-1}$ and assume that $g(u) \to 0$ as $u \to \partial D$ (the boundary as a subset of $\mathbb{R}^{n-1}$). It is convenient to set $g(u) = 0$ for $u \notin D$, and then g is defined and continuous on all $\mathbb{R}^{n-1}$. Set

$$\Omega = \{(u, v) \in \mathbb{R}^n : u \in D, |v| < g(u)\}, \tag{2.1}$$

and for any subset $A \subset \mathbb{R}^n$, $A^+ = A \cap (\mathbb{R}^n)^+$.

The normal line of $(\partial\Omega)^+$ at a point $(u, g(u))$ is given in parametrized form as

$$t \mapsto (u, g(u)) + t(\nabla g(u), -1))$$

and it intersects the hyperplane $\mathbb{R}^{n-1}$ for $t = g(u)$, i.e., at the *foot point*

$$p(u) = u + g(u)\nabla g(u).$$

Let

$$N_u = \{(u, g(u)) + t(\nabla g(u), -1)) : 0 < t < g(u)\}$$

be the part of the normal line which is between the base point on $(\partial\Omega)^+$ and the foot point. The length of N_u is

$$|N_u| = g(u)\sqrt{1 + |\nabla g(u)|^2}.$$

For any point $x \in \Omega^+$ there is at least one closest point $(u, g(u))$ on $(\partial\Omega)^+$. Then $x \in N_u$ and, in particular, $x \in B(p(u), |N_u|)$. Note that N_u is one of the radii in $B(p(u), |N_u|)$. It follows that we always have inclusions

$$\Omega^+ \subset \bigcup_{u \in D} N_u \subset \bigcup_{u \in D} B(p(u), |N_u|).$$

Also,

$$\Omega \subset \bigcup_{u \in D} B(p(u), |N_u|). \tag{2.2}$$

2.2. The first and second fundamental forms

We shall discuss $(\partial\Omega)^+$ from a differential geometric point of view. We consider it as a Riemannian manifold of dimension $n - 1$ and with coordinates $u_1, ..., u_{n-1}$. The Riemannian metric is that inherited from $\mathbb{R}^n$. We shall write certain quantities considered as vectors in $\mathbb{R}^n$ in bold. These involve the "moving point" on $(\partial\Omega)^+$ (or lift map $D \to (\partial\Omega)^+$)

$$\mathbf{x} = \mathbf{x}(u) = (u, g(u)),$$

its differential (a vector-valued differential form)

$$d\mathbf{x} = (du_1, ..., du_{n-1}, \sum_{i=1}^{n-1} \frac{\partial g(u)}{\partial u_i} du_i),$$

the unit normal vector

$$\mathbf{n} = \mathbf{n}(u) = \frac{(\nabla g(u), -1)}{\sqrt{|\nabla g(u)|^2 + 1}}$$

and its differential $d\mathbf{n}$ (which becomes somewhat complicated when written out in components). For a point $x \in \Omega^+$ we sometimes write it in bold if we think of it as the vector from the origin to x.

A tangent vector $\boldsymbol{\xi}$ on $(\partial\Omega)^+$ may be thought of in an abstract way as a derivation:

$$\boldsymbol{\xi} = \sum_{i=1}^{n-1} \xi_i \frac{\partial}{\partial u_i}.$$

Letting this $\boldsymbol{\xi}$ act on the moving point $\mathbf{x}(u)$ or, equivalently, letting the differential $d\mathbf{x}$ act on $\boldsymbol{\xi}$ gives the same tangent vector regarded as a vector embedded in $\mathbb{R}^n$:

$$\langle d\mathbf{x}, \boldsymbol{\xi}\rangle = (\xi_1, ..., \xi_{n-1}, \sum_{i=1}^{n-1} \xi_i \frac{\partial g}{\partial u_i}).$$

In classical differential geometry (see for example [3]) one associates to any hypersurface in Euclidean space two fundamental forms. The first fundamental form is the metric tensor, which gives the inner product on each tangent space. It is in our case

$$\begin{aligned} ds^2 = d\mathbf{x} \cdot d\mathbf{x} &= du_1^2 + ... + du_{n-1}^2 + (\sum_{i=1}^{n-1} \frac{\partial g}{\partial u_i} du_i)^2 \\ &= \sum_{i,j=1}^{n-1} (\delta_{ij} + \frac{\partial g}{\partial u_i}\frac{\partial g}{\partial u_j}) du_i \otimes du_j, \end{aligned}$$

where the dot denotes the scalar product in $\mathbb{R}^n$. The second fundamental form is (up to a sign)

$$d\mathbf{x} \cdot d\mathbf{n} = \frac{1}{\sqrt{1 + |\nabla g|^2}} \sum_{i,j=1}^{n-1} \frac{\partial^2 g}{\partial u_i \partial u_j} du_i \otimes du_j.$$

It contains information on how $\mathbf{n}$ rotates in different directions, i.e. on how $(\partial\Omega)^+$ is curved within $\mathbb{R}^n$. The above expression for $d\mathbf{x} \cdot d\mathbf{n}$ can be derived as follows: Since $\mathbf{n}$ is orthogonal to $(\partial\Omega)^+$ we have $\frac{\partial \mathbf{x}}{\partial u_i} \cdot \mathbf{n} = 0$ for all i. Therefore

$$\begin{aligned} \frac{\partial \mathbf{x}}{\partial u_i} \cdot \frac{\partial \mathbf{n}}{\partial u_j} &= \frac{\partial}{\partial u_j}(\frac{\partial \mathbf{x}}{\partial u_i} \cdot \mathbf{n}) - \frac{\partial^2 \mathbf{x}}{\partial u_i \partial u_j} \cdot \mathbf{n} = 0 - (0, \frac{\partial^2 g}{\partial u_i \partial u_j}) \cdot \mathbf{n} \\ &= \frac{1}{\sqrt{1 + |\nabla g|^2}} \frac{\partial^2 g}{\partial u_i \partial u_j}, \end{aligned}$$

proving the formula.

Both fundamental forms are symmetric covariant 2-tensors, i.e., symmetric bilinear forms on each tangent space. When considered as bilinear forms we shall

call them A and B respectively (namely $A = d\mathbf{x} \cdot d\mathbf{x}$, $B = d\mathbf{x} \cdot d\mathbf{n}$). The above formulas then mean that for (abstract) tangent vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ we have

$$A(\boldsymbol{\xi}, \boldsymbol{\eta}) = \sum_{i=1}^{n-1} \xi_i \eta_i + \sum_{i,j=1}^{n-1} \frac{\partial g}{\partial u_i} \frac{\partial g}{\partial u_j} \xi_i \eta_j,$$

$$B(\boldsymbol{\xi}, \boldsymbol{\eta}) = \frac{1}{\sqrt{1 + |\nabla g|^2}} \sum_{i,j=1}^{n-1} \frac{\partial^2 g}{\partial u_i \partial u_j} \xi_i \eta_j.$$

Note that A is positive definite.

The principal curvatures of $(\partial\Omega)^+$ and the corresponding principal directions are the eigenvalues and eigendirections of the derivative of the Gauss normal map, taking any point $\mathbf{x} \in (\partial\Omega)^+$ onto the normal vector at the point: $\mathbf{n}(\mathbf{x}) \in S^{n-1}$; the derivative is the induced linear map between the tangent spaces, intuitively $d\mathbf{x} \mapsto d\mathbf{n}$. In terms of abstract tangent vectors this becomes $C : \boldsymbol{\xi} \mapsto \boldsymbol{\eta}$, where $\langle d\mathbf{x}, \boldsymbol{\eta} \rangle = \langle d\mathbf{n}, \boldsymbol{\xi} \rangle$, and one can also say that it is the map obtained when expressing the second fundamental form in terms of the first:

$$A(\boldsymbol{\zeta}, C\boldsymbol{\xi}) = B(\boldsymbol{\zeta}, \boldsymbol{\xi}).$$

It is well-known (and easy to prove) that these eigenvalues and eigendirections coincide with the stationary points and corresponding stationary directions for the quotient of the fundamental forms, namely for the map

$$\boldsymbol{\xi} \mapsto \frac{B(\boldsymbol{\xi}, \boldsymbol{\xi})}{A(\boldsymbol{\xi}, \boldsymbol{\xi})}$$

($\boldsymbol{\xi} \in \mathbb{R}^{n-1}$, $\boldsymbol{\xi} \neq 0$). For example, the smallest eigenvalue (the smallest principal curvature) coincides with the minimum value of B/A.

2.3. The function Φ

Next we introduce the function

$$\Phi(u) = \frac{1}{2}(|u|^2 + g(u)^2) = \frac{1}{2}|\mathbf{x}(u)|^2, \quad (u \in D)$$

namely half of the squared distance from points on $(\partial\Omega)^+$ to the origin. More generally, for any $c = (a, b)$ with $a \in \mathbb{R}^{n-1}$, $b \geq 0$ we set

$$\Phi_c(u) = \frac{1}{2}(|u - a|^2 + (g(u) - b)^2) = \frac{1}{2}|\mathbf{x}(u) - c|^2,$$

considered to be defined for those $u \in D$ for which $g(u) > b$. Then Φ_c is twice continuously differentiable with

$$\frac{\partial \Phi_c}{\partial u_i} = u_i - a_i + (g(u) - b)\frac{\partial g}{\partial u_i},$$

$$\frac{\partial^2\Phi_c}{\partial u_i\partial u_j} = \delta_{ij} + \frac{\partial g}{\partial u_i}\frac{\partial g}{\partial u_j} + (g(u)-b)\frac{\partial^2 g}{\partial u_i\partial u_j} \tag{2.3}$$

$$= \frac{\partial^2\Phi}{\partial u_i\partial u_j} - b\frac{\partial^2 g}{\partial u_i\partial u_j}. \tag{2.4}$$

For $c=0$ we can write

$$\frac{\partial^2\Phi}{\partial u_i\partial u_j} = \delta_{ij} + \frac{\partial g}{\partial u_i}\frac{\partial g}{\partial u_j} + g\sqrt{1+|\nabla g|^2}\frac{1}{\sqrt{1+|\nabla g|^2}}\frac{\partial^2 g}{\partial u_i\partial u_j}.$$

Thus considering the Hessian matrix $\nabla^2\Phi = (\frac{\partial^2\Phi}{\partial u_i\partial u_j})$ as a tensor, or bilinear form, we see that it is related to the two fundamental forms by

$$\nabla^2\Phi = d\mathbf{x}\cdot d\mathbf{x} + |N_u| d\mathbf{x}\cdot d\mathbf{n} = A + |N_u|B. \tag{2.5}$$

We finally notice that Φ is related to the foot point map p by

$$\nabla\Phi = p. \tag{2.6}$$

More generally, for any $a\in\mathbb{R}^{n-1}$ we have

$$\nabla\Phi_{(a,0)}(u) = p(u) - a. \tag{2.7}$$

2.4. The Poincaré metric

The Poincaré metric in $(\mathbb{R}^n)^+$ is given by

$$ds^2 = \frac{1}{v^2}(\sum_{j=1}^{n-1} du_j^2 + dv^2)$$

The geodesics with respect to the Poincaré metric are the vertical straight lines

$$\begin{cases} u = \text{constant}, \\ v > 0 \end{cases}$$

together with all vertical semicircles with centers on $\mathbb{R}^{n-1}$, namely all curves of the form

$$\begin{cases} u\in L,\ v>0, \\ |u-a|^2+v^2 = r^2, \end{cases} \tag{2.8}$$

where L is a straight line in $\mathbb{R}^{n-1}$, $a\in L$ and $r>0$.

We consider now the variable transformation

$$T : (u,v)\mapsto (s,t),$$

where $s\in\mathbb{R}^{n-1}$, $t\in\mathbb{R}$, defined by

$$\begin{cases} s = u, \\ t = \frac{1}{2}(|u|^2+v^2) \end{cases}$$

and, conversely,

$$\begin{cases} u = s, \\ v = \sqrt{2t-|s|^2}. \end{cases} \tag{2.9}$$

Thus T is one-to-one and takes $(\mathbb{R}^n)^+$ in the (u, v)-space onto the epiparabola

$$W = \{(s, t) \in \mathbb{R}^n : t > \frac{1}{2}|s|^2\}$$

in the (s, t)-space. By the definition of Φ, T maps Ω^+ onto the set

$$V = \{(s, t) \in W : s \in D,\, t < \Phi(s)\},$$

and it maps the graph of g onto the graph of Φ. If Φ is extended to all $\mathbb{R}^{n-1}$ by setting $\Phi(u) = \frac{1}{2}|u|^2$ outside D (this corresponds to extending g by zero outside D) then $W \setminus V$ is the epigraph of Φ.

Lemma 2.1. *The above map*

$$T : (\mathbb{R}^n)^+ \to W$$

gives a one-to-one correspondence between the set of geodesics with respect to the Poincaré metric in $(\mathbb{R}^n)^+$ *and the set of straight lines in* W.

Proof. Let γ be a geodesic in $(\mathbb{R}^n)^+$. If γ is a vertical line, then also $T(\gamma)$ is a vertical line in W, hence a straight line. If γ is of the form (2.8), then just performing the substitution (2.9) we see that $T(\gamma)$ becomes

$$\begin{cases} s \in L, \\ t = s \cdot a + \frac{1}{2}(r^2 - |a|^2), \end{cases}$$

hence it is a straight line. And the arguments can easily be run in the other direction: every straight line in the (s, t)-space which has a nonempty intersection with W is of the form $T(\gamma)$ for some geodesic γ in $(\mathbb{R}^n)^+$. $\square$

In the new coordinates $(s_1, \ldots, s_{n-1}, t)$ the Poincaré metric takes the form

$$ds^2 = \frac{1}{(2t - |s|^2)^2}\big((2t - |s|^2)\sum_{j=1}^{n-1}(ds_j)^2 + (\sum_{j=1}^{n-1} s_j ds_j)^2 -$$

$$-\sum_{j=1}^{n-1} s_j(ds_j \otimes dt + dt \otimes ds_j) + (dt)^2\big),$$

as an easy calculation shows. Hence this is a metric in W which is complete and has constant negative curvature, and whose geodesics are Euclidean straight lines.

3. Equivalent criteria for the inner ball condition

Below we state equivalent criteria for the domain Ω defined by (2.1) to satisfy the inner ball condition for balls centred on the symmetry plane. The definition of the inner ball condition can be taken to be statement (v) in the theorem.

Theorem 3.1. *With assumptions and notations as in Section 2 the following statements are equivalent.*

(i) *The restriction of* Φ *to any convex subdomain of* D *is convex. (Note that* D *is not required to be convex itself.) Equivalent formulations: the matrix of second derivatives is positive semidefinite:* $\nabla^2\Phi \geq 0$ *in* D*; the extension of* Φ *to* $\mathbb{R}^{n-1}$ *is convex;* $W \setminus V$ *is convex as a set.*

(ii) *For every* $c = (a, b)$ *with* $a \in \mathbb{R}^{n-1}$, $b > 0$*, the function* Φ_c *is convex (alternatively: strictly convex) when restricted to any convex subdomain of the set of* $u \in D$ *for which* $g(u) > b$*.*

(iii) *The restriction of the foot point map* p *to any convex subdomain of* D *is monotone, i.e.*

$$(p(u) - p(u')) \cdot (u - u') \geq 0$$

for all u, u' *in the subdomain. (Here the dot denotes the scalar product in* $\mathbb{R}^{n-1}$*.) Equivalently, the matrix* $(\frac{\partial p_i}{\partial u_j} + \frac{\partial p_j}{\partial u_i})$ *is positive semidefinite.*

(iv)

$$\Omega = \bigcup_{u \in D} B(p(u), |N_u|).$$

(v) *There exist radii* $r = r(u) > 0$ *such that*

$$\Omega = \bigcup_{u \in D} B(u, r(u)).$$

(vi)

$$N_u \cap N_{u'} = \emptyset \quad \text{for } u \neq u' \quad (u, u' \in D).$$

(vii) *The principal curvatures at any point* $(u, g(u))$ *of* $(\partial\Omega)^+$ *are* $\geq -\frac{1}{|N_u|}$*.*

(viii) *Every point in* Ω^+ *has a unique closest neighbor on* $(\partial\Omega)^+$*.*

(ix) *Every point on* $(\partial\Omega)^+$ *is a closest point on* $\partial\Omega$ *for some point in* D*.*

(x) $(\mathbb{R}^n)^+ \setminus \Omega$ *is convex with respect to the Poincaré metric in* $(\mathbb{R}^n)^+$*.*

Proof. (i)⇒(ii): Fix $b > 0$ and $u \in D$. For any $\boldsymbol{\xi} \in \mathbb{R}^{n-1}$ with $|\boldsymbol{\xi}| = 1$, let

$$\alpha = \sum_{i,j=1}^{n-1} \frac{\partial^2 g}{\partial u_i \partial u_j} \xi_i \xi_j.$$

If $\alpha \geq -\frac{1}{2(g(u)-b)}$ then it follows from (2.3) that

$$\sum_{i,j=1}^{n-1} \frac{\partial^2 \Phi_c}{\partial u_i \partial u_j} \xi_i \xi_j \geq \frac{1}{2},$$

while if $\alpha < -\frac{1}{2(g(u)-b)}$ equation (2.4) shows that

$$\sum_{i,j=1}^{n-1} \frac{\partial^2 \Phi_c}{\partial u_i \partial u_j} \xi_i \xi_j \geq \frac{b}{2(g(u) - b)}.$$

From these two inequalities the strict convexity of Φ_c follows. (We even get a uniform lower bound of $\nabla^2\Phi_c$ in $\{u \in D : g(u) > b\}$.)

(ii)⇒(i): Just let $b \to 0$ in (2.4).

(i)⇔(iii): This follows from (2.6), which also shows that the matrix $(\frac{\partial p_i}{\partial u_j})$ is symmetric itself $(\frac{\partial p_i}{\partial u_j}=\frac{\partial^2\Phi}{\partial u_i\partial u_j})$.

(i)⇔(vii): As remarked in Subsection 2.2 the smallest principal curvature of $(\partial\Omega)^+$ coincides with the minimum value of the quotient B/A between the two fundamental forms. Therefore it is immediate from (2.5) that (vii) is equivalent to $\nabla^2\Phi\geq 0$, i.e., to (i).

(i)⇒ (iv): By (2.2) we only need to show that

$$B(p(u),|N_u|)\subset\Omega. \tag{3.1}$$

for all $u\in D$.

Fix any $u_0\in D$. Then taking $a=p(u_0)$ in (2.7) we see that the map $u\mapsto \Phi_{(p(u_0),0)}(u)$ has a stationary point at $u=u_0$. In view of the interpretation of $\Phi_{(p(u_0),0)}$ as half of the squared distance from $(p(u_0),0)$ to $(u,g(u))$ it follows that this stationary point is a (global) minimum if and only if (3.1) holds for $u=u_0$. But when Φ (equivalently $\Phi_{(p(u_0),0)}$) is convex then every stationary point of $\Phi_{(p(u_0),0)}$ is a global minimum. Hence (3.1) holds for $u=u_0$.

(iv)⇒(v): If the representation in (iv) holds then we get a representation as in (v) by adding small balls $B(u,r(u))\subset\Omega$ for those $u\in D$ which are not in the range of p. (Clearly p maps D into D when (iv) holds, but it need not be onto.)

(v)⇒(iv): Let $(u,g(u))\in(\partial\Omega)^+$. Then, if (v) holds, there exist points $a_j\in D$ and $x_j\in B(a_j,r(a_j))$ such that $x_j\to(u,g(u))$ as $j\to\infty$. The smoothness of $(\partial\Omega)^+$ and the inclusions $B(a_j,r(a_j))\subset\Omega$ imply that $a_j\to p(u)$ and $r(a_j)\to|N_u|$ and we conclude that $B(p(u),|N_u|)\subset\Omega$. Now (iv) follows from (2.2).

(iv)⇔(ix): Note that $(u,g(u))\in(\partial\Omega)^+$ is a closest point of $a\in D$ if and only if $a=p(u)$ and $B(p(u),|N_u|)\subset\Omega$. Thus a is determined by $(u,g(u))$ and it follows immediately (in view also of (2.2)) that all points on $(\partial\Omega)^+$ are such closest points if and only if the inner ball condition in (iv) holds.

(iv)⇒(vi): Assume that (vi) fails, so that there exists a point $x\in N_{u_1}\cap N_{u_2}$ for some $u_1,u_2\in D$, $u_1\neq u_2$. Without loss of generality $|(u_1,g(u_1))-x|\leq |(u_2,g(u_2))-x|$. Then $(u_1,g(u_1))\in\overline{B(x,|(u_2,g(u_2))-x|)}$. Since $x\in N_{u_2}$ we have $\overline{B(x,|(u_2,g(u_2))-x|)}\subset B(p(u_2),|N_{u_2}|)\cup\{(u_2,g(u_2))\}$. It follows that $(u_1,g(u_1))\in B(p(u_2),|N_{u_2}|)$. But $(u_1,g(u_1))\notin\Omega$, hence (iv) does not hold.

(vi)⇒(viii): If $x\in\Omega^+$ has two closest neighbours $(u_1,g(u_1)),(u_2,g(u_2))\in(\partial\Omega)^+$ then $x\in N_{u_1}\cap N_{u_2}$.

(viii)⇒(ii): This conclusion is somewhat related to [9], Theorem 2.1.30 (attributed to Motzkin), but we give an independent proof.

Assume that (ii) fails and we shall produce a point $c\in\Omega^+$ with at least two closest neighbors on $(\partial\Omega)^+$. We may assume that $\Phi_{(0,b)}$ is not convex for some

$b > 0$ in a convex subdomain of $D_b = \{u \in D : g(u) > b\}$. We extend $\Phi_{(0,b)}$ to all $\mathbb{R}^{n-1}$ by setting

$$\begin{aligned}\tilde{\Phi}_{(0,b)} &= \begin{cases}\Phi_{(0,b)} & \text{for } u \in D_b \\ \frac{1}{2}|u|^2 & \text{for } u \in \mathbb{R}^{n-1} \setminus D_b\end{cases} \\ &= \frac{1}{2}(|u|^2 + |(g(u)-b)_+|^2).\end{aligned}$$

In the latter expression the plus subscript denotes positive part and g is assumed to be extended by zero outside D.

The function $\tilde{\Phi}_{(0,b)}$ is easily seen to be continuously differentiable and by assumption it is not convex. Let Ψ be the convex envelope of $\tilde{\Phi}_{(0,b)}$, which can be defined as

$$\Psi(u) = \sup\{L(u) : L \text{ is affine and } \leq \tilde{\Phi}_{(0,b)}\}. \tag{3.2}$$

Equivalently, Ψ is the function whose epigraph, $\operatorname{epi}\Psi = \{(u,v) \in \mathbb{R}^n : v \geq \Psi(u)\}$, is the closed convex hull of the epigraph of $\tilde{\Phi}_{(0,b)}$. In the present case, because of the strict convexity of $\tilde{\Phi}_{(0,b)}$ outside a compact set, the convex hull will be closed right away (before taking the closure). Thus

$$\operatorname{epi}\Psi = \operatorname{conv}(\operatorname{epi}\tilde{\Phi}_{(0,b)}), \tag{3.3}$$

conv denoting "convex hull".

Since $\tilde{\Phi}_{(0,b)}$ is not convex there exists $u_0 \in \mathbb{R}^{n-1}$ such that

$$\Psi(u_0) < \tilde{\Phi}_{(0,b)}(u_0).$$

It is easy to see that the supremum in (3.2) is attained for each fixed u. Hence

$$\Psi(u_0) = L(u_0)$$

for some affine L satisfying

$$L(u) \leq \tilde{\Phi}_{(0,b)}(u) \quad \text{for all } u. \tag{3.4}$$

By (3.3) the point $(u_0, L(u_0)) = (u_0, \Psi(u_0)) \in \operatorname{epi}\Psi$ can be written as a finite convex combination of points in the epigraph of $\tilde{\Phi}_{(0,b)}$. Since this convex combination must take place within the graph of L, u_0 is a convex combination of finitely many points u (at least two are needed) for which $L(u) = \tilde{\Phi}_{(0,b)}(u)$. It follows that equality is attained in (3.4) for at least two different u.

Now write

$$L(u) = a \cdot u + k,$$

where $a \in \mathbb{R}^{n-1}$, $k \in \mathbb{R}$ and the dot denotes scalar product in $\mathbb{R}^{n-1}$. Then (3.4) can be written

$$|u-a|^2 + |(g(u)-b)_+|^2 \geq 2k + |a|^2, \tag{3.5}$$

where equality is attained for at least two different $u \in \mathbb{R}^{n-1}$.

Since the left member in (3.5) is ≥ 0 with equality in at most one point (namely $u = a$) it follows that

$$2k + |a|^2 > 0. \tag{3.6}$$

This positive minimum value can not be attained when $g(u) \leq b$ because for those u the derivative of the left member in (3.5) is $2(u-a) \neq 0$. Note that $u = a$ does not satisfy (3.5) when $g(u) \leq b$ because of (3.6).

Thus $g(u) > b$ whenever equality holds in (3.5). Clearly also $g(a) > b$ by (3.5) and (3.6), showing that $a \in D_b$. It now follows that

$$|u-a|^2 + |g(u)-b|^2 \geq 2k + |a|^2$$

for all $u \in D$ with equality for at least two different u satisfying $g(u) > b$.

In conclusion we have produced a point $c = (a, b) \in \Omega^+$ having at least two closest neighbors on $(\partial\Omega)^+$.

(i)⇔(x): That $(\mathbb{R}^n)^+ \setminus \Omega$ is convex with respect to the Poincaré metric means by definition that every Poincaré geodesic γ in $(\mathbb{R}^n)^+$ meets Ω at most once (i.e., $\gamma \setminus \Omega$ has at most one component). By Lemma 2.1 this occurs if and only if the straight line $T(\gamma)$ in W meets V at most once. This is the same as saying that $W \setminus V$ is convex as a set, which is easily seen to be equivalent to the convexity of Φ ($W \setminus V$ is the epigraph of the extension of Φ obtained by setting $g = 0$ outside D). This proves (i)⇔(x).

□

4. Boundaries of quadrature domains

4.1. Notations

In the rest of the paper we shall write points in $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ typically as $x = (x', x_n)$, with $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. The letter u will be used to denote a specific function, see below.

Let μ be a finite positive (and not identically zero) Borel measure with compact support in the hyperplane $\mathbb{R}^{n-1} = \{x \in \mathbb{R}^n : x_n = 0\}$. Then there exists a unique quadrature open set $\Omega \subset \mathbb{R}^n$ for subharmonic functions with respect to μ, see [10], [6]. Thus $\mu(\mathbb{R}^n \setminus \Omega) = 0$ and

$$\int_\Omega h\, d\mu \leq \int_\Omega h\, dm$$

for all integrable (with respect to Lebesgue measure m) subharmonic functions h in Ω. By uniqueness Ω is symmetric with respect to the hyperplane $\mathbb{R}^{n-1}$. We shall assume that Ω is connected (i.e., a quadrature domain) and discuss the curvature of $(\partial\Omega)^+$. In the nonconnected case the discussions will simply apply to each component of Ω.

Let

$$u = U^\mu - U^\Omega$$

be the difference between the Newtonian potentials of μ and Ω, the latter considered as a body of density one. Then

$$\begin{cases} u \geq 0 \quad \text{in } \mathbb{R}^n, \\ u = 0 \quad \text{outside } \Omega, \\ \Delta u = \chi_\Omega - \mu \quad \text{in } \mathbb{R}^n. \end{cases} \tag{4.1}$$

From (4.1) it follows that $u \in C^1((\mathbb{R}^n)^+)$, hence $\nabla u = 0$ on $(\mathbb{R}^n)^+ \setminus \Omega$ (since u attains its minimum value there).

For proving the inner ball property it is enough to consider the case that μ has a continuous density on $\mathbb{R}^{n-1}$:

$$d\mu = f dx_1 \dots dx_{n-1}, \tag{4.2}$$

with $f = f(x') > 0$ on $D = \Omega \cap \mathbb{R}^{n-1}$, $f = 0$ on $\mathbb{R}^{n-1} \setminus D$ and f continuous on $\mathbb{R}^{n-1}$ (see Subsection 4.5 for a motivation of this). In that case, u is continuous across $\mathbb{R}^{n-1}$ while $\frac{\partial u}{\partial x_n}$ makes a jump of size f across $\mathbb{R}^{n-1}$ (this is the meaning of the distributional Laplacian in (4.1)). In summary, assuming (4.2) u satisfies in Ω^+ the overdetermined system

$$\begin{cases} \Delta u = 1 \quad \text{in} \quad \Omega^+, \\ u = |\nabla u| = 0 \quad \text{on} \quad (\partial\Omega)^+, \\ -2\frac{\partial u}{\partial x_n} = f \quad \text{on} \quad D, \end{cases} \tag{4.3}$$

where in the last equation $\frac{\partial u}{\partial x_n}$ should be understood as boundary values from the upper half space. If μ is not of the form (4.2), the system (4.3) still holds with the last equation appropriately reformulated.

Returning to the general case, the components of ∇u are harmonic functions in Ω^+. By applying the maximum principle to $\frac{\partial u}{\partial x_n}$ it follows that

$$\begin{aligned} u > 0 \quad &\text{in } \Omega^+, \\ \frac{\partial u}{\partial x_n} < 0 \quad &\text{in } \Omega^+. \end{aligned} \tag{4.4}$$

Thus u is strictly decreasing with respect to x_n, showing that $(\partial\Omega)^+$ is the graph of a function, i.e., is of the form $x_n = g(x')$ for some positive function g defined in $D = \Omega \cap \mathbb{R}^{n-1}$. With a more detailed analysis one can show [4], [1], [6] that the function g is actually real analytic and tends to zero at ∂D. It follows that we are in the setting of Sections 2 and 3.

The aim of the forthcoming analysis is to approach the following conjecture.

Conjecture 4.1. *Any subharmonic quadrature domain Ω as above satisfies the equivalent conditions in Theorem 3.1.*

As have already been remarked, Conjecture 4.1 has been settled in the case $n = 2$. Two completely different proofs appear in [7] and [8]. A third proof, perhaps somewhat related to the one in [8], will be given in the present paper (Corollary 4.9).

4.2. The function ρ and the 2-form ω

Let (r, θ) be spherical coordinates in $\mathbb{R}^n$ in the sense that $r = |\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_n^2}$, $\theta = \frac{\mathbf{x}}{|\mathbf{x}|} \in S^{n-1}$ (if $\mathbf{x} \neq 0$). Then $r\frac{\partial}{\partial r} = \mathbf{x} \cdot \nabla = x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n}$. In order to analyse the curvature of $(\partial\Omega)^+$ we introduce a function ρ and a 2-form ω according to

$$\rho = \frac{1}{2}r^2 - r\frac{\partial u}{\partial r} - (n-2)u, \qquad (4.5)$$
$$\omega = rdr \wedge du = d(\frac{1}{2}r^2 du).$$

Thus,

$$\omega = \sum_{i<j} \omega_{ij} dx_i \wedge dx_j = \sum_{i,j=1}^{n} \omega_{ij} dx_i \otimes dx_j$$

$(dx_i \wedge dx_j = dx_i \otimes dx_j - dx_j \otimes dx_i)$ with

$$\omega_{ij} = x_i \frac{\partial u}{\partial x_j} - x_j \frac{\partial u}{\partial x_i}. \qquad (4.6)$$

Note that the restriction of ρ to $(\partial\Omega)^+$ coincides with the function Φ in Subsection 2.3 when the latter is regarded as a function of $x' = (x_1, \ldots, x_{n-1})$: with $(\partial\Omega)^+$ given by $x_n = g(x')$ we have

$$\Phi(x') = \rho(x', g(x')).$$

Recall that the Hodge star operator [3] is defined on basic forms

$$\alpha = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

as

$$*\alpha = \pm dx_{i_{p+1}} \wedge \cdots \wedge dx_{i_n},$$

where the sign is chosen so that

$$\alpha \wedge *\alpha = dx_1 \wedge \cdots \wedge dx_n = dx \qquad \text{(the volume form)}$$

and $(i_1, \ldots, i_n)$ denotes any permutation of $(1, \ldots, n)$. Interior multiplication of a p-form α with a vector field $\boldsymbol{\xi}$ is the $(p-1)$-form obtained by letting $\boldsymbol{\xi}$ occupy the first position when α is viewed as an antisymmetric operator acting on p vector fields:

$$(i(\boldsymbol{\xi})\alpha)(\boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_p) = \alpha(\boldsymbol{\xi}, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_p).$$

We shall also need the Lie derivative $L_{\boldsymbol{\xi}}$ with respect to a vector field $\boldsymbol{\xi}$, and recall that its action on any p-form α is

$$L_{\boldsymbol{\xi}}(\alpha) = i(\boldsymbol{\xi})d\alpha + d(i(\boldsymbol{\xi})\alpha),$$

and that its action on a vector field $\boldsymbol{\eta}$ is the commutator

$$L_{\boldsymbol{\xi}}\boldsymbol{\eta} = [\boldsymbol{\xi}, \boldsymbol{\eta}],$$

when all vector fields are considered as differential operators ($\boldsymbol{\xi}$ corresponds to the directional derivative $\boldsymbol{\xi} \cdot \nabla$ etc.).

In our Euclidean setting of differential geometry there are natural one-to-one correspondences between vector fields, 1-forms and $(n-1)$-forms. The basis elements correspond to each other according to

$$\mathbf{e}_i \leftrightarrow dx_i \leftrightarrow *dx_i,$$

and the forms/vectors simply keep their coefficients with respect to the above bases under the identifications. This means for example that if $\mathbf{a}, \mathbf{b}$ are vectors, α a 1-form and β an $(n-1)$-form, then

$$\mathbf{a} \leftrightarrow \alpha \quad \text{if and only if} \quad i(\mathbf{a})dx = *\alpha,$$

$$\mathbf{b} \leftrightarrow \beta \quad \text{if and only if} \quad i(\mathbf{b})dx = \beta.$$

The definition of ω can be expressed in terms of interior multiplication and the Hodge star operator as

$$*\omega = -i(r\frac{\partial}{\partial r})(*du).$$

4.3. The Cauchy-Riemann system

The fundamental relationship between ρ and ω is expressed in the following Cauchy-Riemann type system.

Theorem 4.2. *We have*

$$* \, d\rho = d(*\omega) \quad \text{in} \quad \Omega^+. \tag{4.7}$$

Written out in components this is

$$\frac{\partial \rho}{\partial x_k} = \sum_{j=1}^{n} \frac{\partial \omega_{kj}}{\partial x_j} \quad (k = 1, \ldots, n). \tag{4.8}$$

Proof. It is easy to check that (4.7), when spelled out, simply becomes (4.8), so we just verify (4.8): Differentiating (4.6) we get

$$\frac{\partial \omega_{kj}}{\partial x_j} = x_k \frac{\partial^2 u}{\partial x_j^2} - \frac{\partial u}{\partial x_k} - x_j \frac{\partial^2 u}{\partial x_j \partial x_k}$$

when $j \neq k$. For $j = k$ there is an additional term $\frac{\partial u}{\partial x_j}$, which cancels the term $-\frac{\partial u}{\partial x_k}$. Summing over j therefore gives

$$\sum_{j=1}^{n} \frac{\partial \omega_{kj}}{\partial x_j} = x_k \Delta u - (n-1)\frac{\partial u}{\partial x_k} - r\frac{\partial}{\partial r}(\frac{\partial u}{\partial x_k}).$$

On the other hand, differentiating (4.5) gives

$$\frac{\partial \rho}{\partial x_k} = x_k - (n-2)\frac{\partial u}{\partial x_k} - \frac{\partial}{\partial x_k}(r\frac{\partial u}{\partial r}).$$

By a change of order of differentiating, and in view of $\Delta u = 1$ in Ω^+, (4.8) now follows. $\square$

Introducing the coexterior derivative δ, defined on p-forms by

$$\delta = (-1)^{n(p+1)+1} * d*,$$

the system (4.7) can be written

$$d\rho = \delta\omega. \tag{4.9}$$

Trivially $\delta\rho = 0$, and since ω by definition is exact, $d\omega = 0$. Recall that the Hodge Laplacian is

$$\Delta = -(\delta d + d\delta)$$

and that it in our Euclidean setting simply equals the ordinary Laplacian acting on the coefficients of the forms it is applied to. From (4.9) we find (using $\delta \circ \delta = 0$ etc.)

Corollary 4.3.

$$\begin{cases} \delta\rho = 0, \\ \delta d\rho = 0, \end{cases} \qquad \begin{cases} d\omega = 0, \\ d\delta\omega = 0, \end{cases}$$

in particular

$$\Delta\rho = 0, \quad \Delta\omega = 0.$$

4.4. The vector field $\boldsymbol{\xi}$

Next we introduce the vector field

$$\begin{aligned} \boldsymbol{\xi} = \nabla\rho &= \mathbf{x} - (n-2)\nabla u - \nabla(\mathbf{x}\cdot\nabla u) \\ &= \mathbf{x} - (n - 1 + r\frac{\partial}{\partial r})\nabla u. \end{aligned}$$

For any function φ,

$$i(\nabla\varphi)dx = *d\varphi,$$

hence the Cauchy-Riemann system (4.7) can be written in terms of $\boldsymbol{\xi}$ and ω as

$$i(\boldsymbol{\xi})dx = d(*\omega).$$

Immediate consequences, in view of $i(\boldsymbol{\xi}) \circ i(\boldsymbol{\xi}) = 0$, are

$$\begin{aligned} i(\boldsymbol{\xi})d(*\omega) &= 0, \\ L_{\boldsymbol{\xi}}(*\omega) &= d(i(\boldsymbol{\xi})(*\omega)). \end{aligned} \tag{4.10}$$

When $n = 2$, $i(\boldsymbol{\xi})(*\omega) = 0$ (because it is an $(n-3)$-form), hence $L_{\boldsymbol{\xi}}(*\omega) = 0$.

We proceed with a geometric interpretation of $\boldsymbol{\xi}$ on $(\partial\Omega)^+$.

Proposition 4.4. *At any point $\mathbf{x} \in (\partial\Omega)^+$, $\boldsymbol{\xi} = \nabla\rho$ equals the orthogonal projection of the vector $\mathbf{x}$ onto the tangent space of $(\partial\Omega)^+$ at $\mathbf{x}$. In particular, $\boldsymbol{\xi}$ is tangent to $(\partial\Omega)^+$, i.e., $\frac{\partial\rho}{\partial n} = 0$ on $(\partial\Omega)^+$.*

Proof. Since all components of $*\omega$ vanish on $(\partial\Omega)^+$, the $(n-1)$-form $i(\boldsymbol{\xi})dx = d(*\omega)$ vanishes along $(\partial\Omega)^+$ (i.e., its integral over any piece of $(\partial\Omega)^+$ is zero). This means exactly that $\boldsymbol{\xi}$ is tangent to $(\partial\Omega)^+$. But $\boldsymbol{\xi} = \mathbf{x} - \nabla(\mathbf{x}\cdot\nabla u - (n-2)u)$ and $\nabla(\mathbf{x}\cdot\nabla u + (n-2)u)$ is orthogonal to $(\partial\Omega)^+$ because $\mathbf{x}\cdot\nabla u + (n-2)u = 0$ on $(\partial\Omega)^+$. It follows that $\boldsymbol{\xi}$ is the projection of $\mathbf{x}$ onto the tangent space of $(\partial\Omega)^+$.

□

Corollary 4.5. *A point $\mathbf{x} = (x', x_n) \in (\partial\Omega)^+$ is stationary for $\frac{1}{2}r^2$ on $(\partial\Omega)^+$ equivalently, x' is stationary for Φ, if and only if $\boldsymbol{\xi} = 0$ at $\mathbf{x}$.*

Example 1. In two dimensions, with the identification $\mathbb{R}^2 = \mathbb{C}$ and using (x, y) as coordinates (and with $z = x + \mathrm{i}y$) we can define the Schwarz function in Ω^+ by

$$S(z) = \bar{z} - 4\frac{\partial u}{\partial z}.$$

It is holomorphic in Ω^+ and extends continuously to $(\partial\Omega)^+$ with

$$S(z) = \bar{z} \quad \text{on} \quad (\partial\Omega)^+.$$

In addition, by (4.3),

$$\operatorname{Im} S(x + i0) = f(x) \quad \text{on} \quad \partial(\Omega^+) \cap \mathbb{R}.$$

In the two dimensional case $*\omega$ is a 0-form, i.e., a function, and it is exactly the conjugate harmonic function of ρ. More precisely, ρ, $*\omega$ are the real and imaginary parts of the analytic function $\frac{1}{2}zS(z)$:

$$\frac{1}{2}zS(z) = \frac{1}{2}r^2 - r\frac{\partial u}{\partial r} + \mathrm{i}\frac{\partial u}{\partial \varphi} = \rho + \mathrm{i} * \omega. \tag{4.11}$$

Example 2. In the case of three dimensions we can identify the 2-form ω with the vector field

$$\boldsymbol{\omega} = \mathbf{x} \times \nabla u = \operatorname{curl}\,(\frac{1}{2}r^2\nabla u).$$

The Cauchy-Riemann system $*d\rho = d(*\omega)$ then takes the form

$$\nabla\rho = \operatorname{curl}\,\boldsymbol{\omega}.$$

In terms of $\boldsymbol{\xi} = \nabla\rho$ and $\boldsymbol{\omega}$ we thus have the equations

$$\begin{cases} \operatorname{div}\,\boldsymbol{\omega} = 0, \\ \operatorname{curl}\,\boldsymbol{\xi} = 0, \\ \boldsymbol{\xi} = \operatorname{curl}\,\boldsymbol{\omega}. \end{cases}$$

In particular $\operatorname{div}\,\boldsymbol{\xi} = 0$. Note the triple curl identity $\operatorname{curl}\operatorname{curl}\operatorname{curl}\,(\frac{1}{2}r^2\nabla u) = 0$.

Notice also that the vector field $\boldsymbol{\omega}$ is everywhere tangent to the family of spheres $|\mathbf{x}| = $ constant. Thus, in spherical coordinates (r, θ, φ), where θ (now) is the angle between the vector $\mathbf{x}$ and the x_3-axis and φ is the polar angle for the projection of $\mathbf{x}$ onto the (x_1, x_2)-plane, on writing

$$\boldsymbol{\omega} = \omega_r\mathbf{e}_r + \omega_\theta\mathbf{e}_\theta + \omega_\varphi\mathbf{e}_\varphi$$

we have $\omega_r = 0$. On $D = \Omega \cap \mathbb{R}^2$, i.e., for $\theta = \frac{\pi}{2}$ we have $\omega_\varphi = -r\frac{\partial u}{\partial x_3} = \frac{1}{2}rf(x') > 0$ because $f > 0$ on D.

For the interior multiplication,

$$\begin{aligned} i(\boldsymbol{\xi})(*\omega) = \boldsymbol{\omega}\cdot\boldsymbol{\xi} &= (\mathbf{x}\times\nabla u)\cdot(\mathbf{x} - \nabla(\mathbf{x}\cdot\nabla u) - \nabla u) \\ &= -(\mathbf{x}\times\nabla u)\cdot\nabla(\mathbf{x}\cdot\nabla u). \end{aligned}$$

As to the Lie derivatives of $\boldsymbol{\omega}$ (vector), ω (1-form), $*\omega$ (2-form) we have, when the results are identified with vector fields,

$$\begin{aligned} L_{\boldsymbol{\xi}}(\boldsymbol{\omega}) &= [\boldsymbol{\xi},\boldsymbol{\omega}] = (\boldsymbol{\xi}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{\xi}, \\ L_{\boldsymbol{\xi}}(\omega) &= \text{ as a vector } = \operatorname{curl}(\boldsymbol{\omega}\times\boldsymbol{\xi}) = (\boldsymbol{\xi}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{\xi}, \\ L_{\boldsymbol{\xi}}(*\omega) &= \text{ as a vector } = \nabla(\boldsymbol{\xi}\cdot\boldsymbol{\omega}) = (\boldsymbol{\xi}\cdot\nabla)\boldsymbol{\omega} + (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{\xi}. \end{aligned}$$

We now return to the general case. Since, by (4.4), $\nabla u \neq 0$ in Ω^+, the integral curves of ∇u constitute a smooth (even real analytic) family of curves which fill up Ω^+. The curves are directed downwards ($\nabla u\cdot\mathbf{e}_n < 0$), start immediately inside $(\partial\Omega)^+$ (recall that $\nabla u = 0$ on $(\partial\Omega)^+$) and end up at points of $\operatorname{supp}\mu$. Thus a subdomain $R \subset \Omega^+$ which is bounded by integral curves of ∇u may be thought of as a tube going from $(\partial\Omega)^+$ to $D \subset \mathbb{R}^{n-1}$.

Proposition 4.6. *Let $R \subset \Omega^+$ be a domain such that $\partial R \cap \Omega^+$ is smooth and consists of integral curves of ∇u. Then, assuming $d\mu = f dx_1 \dots dx_{n-1}$ with f continuous on D,*

$$m(R) = \frac{1}{2}\mu(\partial R \cap \mathbb{R}^{n-1}),$$

m denoting Lebesgue measure.

Proof. By assumption, $\nabla u\cdot\mathbf{n} = 0$ on $\partial R \cap \Omega^+$ ($\mathbf{n}$ the normal unit vector on ∂R). Therefore,

$$\begin{aligned} m(R) &= \int_R \Delta u dx = \int_{\partial R} \nabla u\cdot\mathbf{n} d\sigma \\ &= \int_{\partial R\cap\mathbb{R}^{n-1}} \nabla u\cdot\mathbf{n} d\sigma + \int_{\partial R\cap\Omega^+} \nabla u\cdot\mathbf{n} d\sigma + \int_{\partial R\cap(\partial\Omega)^+} \nabla u\cdot\mathbf{n} d\sigma \\ &= \int_{\partial R\cap\mathbb{R}^{n-1}} (-\frac{\partial u}{\partial x_n}) dx_1 \dots dx_{n-1} + 0 + 0 = \frac{1}{2}\mu(\partial R\cap\mathbb{R}^{n-1}). \end{aligned}$$

□

4.5. The curve γ and the map σ

Next we shall study the set where ω vanishes:

$$\gamma = \{x \in \Omega^+ : \omega(x) = 0\}.$$

The vanishing of ω at a point x means by definition that all components of ω vanish. Note that, since $rdr \neq 0$ and $du \neq 0$ in Ω^+, ω vanishes if and only if rdr and du are proportional, i.e., if and only if ∇u is parallel to the vector from x

to the origin, and hence points towards the origin. This is also the same as saying that the projection

$$\nabla_{\partial B_r} u = \nabla u - (\nabla u \cdot \mathbf{e}_r)\mathbf{e}_r$$

of ∇u onto the tangent plane of the sphere $\partial B(0,r)$ vanishes. Hence

$$\begin{aligned}\gamma &= \{x \in \Omega^+ : -\nabla u(x) \text{ is parallel to the vector } \mathbf{x}\} \\ &= \{x \in \Omega^+ : \nabla_{\partial B_r} u(x) = 0\}.\end{aligned} \tag{4.12}$$

Recall that $\mathbf{x}$ denotes the vector from the origin to the point x.

A possible proof of Conjecture 4.1 may be based on establishing that γ is a smooth curve, when nonempty. The strategy would then be as follows. Suppose Ω does not satisfy the inner ball condition, i.e., that the equivalent conditions in Theorem 3.1 do not hold. Then there exists by condition (viii) a point $c = (a,b) \in \Omega^+$ which has at least two closest neighbors on the boundary. Here $a \in \mathbb{R}^{n-1}$, $b > 0$. By "localization" (see [6], [13]) the set $\Omega_c^+ = \{x \in \Omega^+ : x_n > b\}$ will still be the upper half of a quadrature domain, namely for a measure with support on the hyperplane $x_n = b$. Moreover, this measure will have a continuous density function f on $x_n = b$, namely given by

$$f(x_1,\ldots,x_{n-1}) = -2\frac{\partial u}{\partial x_n}(x_1,\ldots,x_{n-1},b)$$

(cf. (4.3)). This is because the localization measure on $x_n = b$ can be obtained from u by keeping u unchanged in $x_n > b$ and defining a new continuation of u to $x_n < b$ by reflection in $x_n = b$. Note by (4.4) that the above f is strictly positive at points belonging to Ω.

Now we translate our system of coordinates so that c becomes the origin in the new coordinate system, and we also adapt the notations in general to fit the new coordinates. In this new situation Ω is a subharmonic quadrature domain for a measure μ in the hyperplane $x_n = 0$ with

$$d\mu = f dx_1 \ldots dx_{n-1}, \tag{4.13}$$

f continuous on $\mathbb{R}^{n-1}$, $f > 0$ on $D = \Omega \cap \mathbb{R}^{n-1}$, and the largest ball $B(0,r_0)$ centered at the origin and contained in Ω touches $(\partial\Omega)^+$ in at least two points. The set γ is expected to meet $(\partial\Omega)^+$ at these points (cf. Proposition 4.10 below), and in addition reach the origin, thus it need to branch. Thus if we can prove that γ is a smooth curve we reach a contradiction. The conclusion then is that the original Ω will have to satisfy the inner ball condition.

Motivated by the above we assume from now on that μ is of the form (4.13) with $f > 0$ on D and continuous on $\mathbb{R}^{n-1}$.

Proposition 4.7. *If $0 \notin \operatorname{conv} \overline{\Omega}$, then γ is empty. If $0 \in \Omega$ then γ intersects every hemisphere $(\partial B(0,r))^+$ with $0 < r < r_0 = \inf_{x \in (\partial\Omega)^+} |\mathbf{x}|$. For $r > 0$ sufficiently small there is exactly one intersection point. At the origin γ starts out in the direction $\boldsymbol{\xi}(0)$. The closure of γ reaches $\Omega \cap \mathbb{R}^{n-1}$ only at the origin.*

Proof. To prove the first statement, assume $\operatorname{supp}\mu \subset \{(x_1,\dots,x_{n-1}) \in \mathbb{R}^{n-1} : x_1 < 0\}$, for example, and consider the component

$$\omega_{1n} = x_1 \frac{\partial u}{\partial x_n} - x_n \frac{\partial u}{\partial x_1}$$

of ω. We have $\Delta\omega_{1n} = 0$ in Ω^+, $\omega_{1n} = 0$ on $(\partial\Omega)^+$. On $\mathbb{R}^{n-1}$, $\omega_{1n} = x_1 \frac{\partial u}{\partial x_n}$, which by assumption (and (4.3)) vanishes for $x_1 \geq 0$ and is nonnegative (and not identically zero) for $x_1 < 0$. Hence $\omega_{1n} > 0$, and so $\omega \neq 0$, in Ω^+.

To prove the second statement, consider any hemisphere $(\partial B(0,r))^+$, $0 < r < r_0$, and the tangent vector field $\nabla_{\partial B_r} u = \nabla u - (\nabla u \cdot \mathbf{e}_r)\mathbf{e}_r$ on it. On the boundary, $\partial B(0,r) \cap \mathbb{R}^{n-1}$, $\nabla_{\partial B_r} u$ points into the lower hemisphere, because $\frac{\partial u}{\partial x_n} < 0$. Thus it follows from well-know index theorems for vector fields (see [3], for example) that $\nabla_{\partial B_r} u$ has to vanish somewhere in $(\partial B(0,r))^+$.

Next, assume $x, y \in \gamma \cap (\partial B(0,r))^+$ $(x \neq y)$. Then $\mathbf{x} = -\lambda \nabla u(x)$, $\mathbf{y} = -\mu \nabla u(y)$ for some $\lambda, \mu > 0$ ($\mathbf{x}$, $\mathbf{y}$ denote x and y considered as vectors). An easy calculation, using $|\mathbf{x}| = |\mathbf{y}|$, shows that

$$\frac{|\nabla u(x) - \nabla u(y)|}{|\mathbf{x} - \mathbf{y}|} = \frac{1}{\lambda} \frac{|\mathbf{x} - \frac{\lambda}{\mu}\mathbf{y}|}{|\mathbf{x} - \mathbf{y}|} \geq \frac{1}{\lambda} \inf_{\alpha > 0} \frac{|\mathbf{x} - \alpha\mathbf{y}|}{|\mathbf{x} - \mathbf{y}|} \geq \frac{1}{2\lambda}.$$

By assumption, ∇u is smooth in Ω^+ up to $\mathbb{R}^{n-1}$ and $|\nabla u| \geq -\frac{\partial u}{\partial x_n} \geq c$ for small $r > 0$ (for some $c > 0$). It follows that $\lambda = \frac{r}{|\nabla u(x)|} \leq \frac{r}{c}$, hence $\lambda \to 0$ if we let $r \to 0$. Thus the difference quotients $\frac{|\nabla u(x) - \nabla u(y)|}{|\mathbf{x} - \mathbf{y}|}$ on $(\partial B(0,r))^+$ tend to infinity as $r \to 0$, which contradicts the smoothness of ∇u up to $\mathbb{R}^{n-1}$. The conclusion is that there cannot be two different points in $\gamma \cap (\partial B(0,r))^+$ for $r > 0$ sufficiently small.

Thus, for small $r > 0$ there is a unique point $x = x(r) \in \gamma \cap (\partial B(0,r))^+$. The direction as $r \to 0$ is (writing $\mathbf{x}(r) = -\lambda(r)\nabla u(x(r))$)

$$\lim_{r \to 0} \frac{\mathbf{x}(r)}{|\mathbf{x}(r)|} = \lim_{r \to 0} \frac{-\lambda(r)(-\nabla u(x(r)))}{\lambda(r)|\nabla u(x(r))|} = -\frac{\nabla u(0)}{|\nabla u(0)|} = \frac{\boldsymbol{\xi}(0)}{|\boldsymbol{\xi}(0)|}.$$

The last statement of the proposition follows directly from the fact that $\omega_{jn} = x_j \frac{\partial u}{\partial x_n} = -\frac{1}{2} x_j f(x') \neq 0$ on $(\Omega \cap \mathbb{R}^{n-1}) \setminus \{0\}$.

$\square$

From the definition (4.6) of ω_{kj} we see that

$$\omega_{kj} = \frac{1}{x_n}(x_j \omega_{kn} - x_k \omega_{jn}),$$

hence ω is algebraically determined by the $n-1$ components $\omega_{1n}, \dots, \omega_{n-1,n}$. It follows that all information of ρ and ω is contained in the map

$$\sigma : \Omega^+ \to \mathbb{R}^n, \quad x \mapsto (\omega_{1n}(x), \dots, \omega_{n-1,n}(x), \rho(x)).$$

In the case of two dimensions this $\sigma = (\omega_{12}, \rho)$ is, with a switch of coordinates, the same as the function $\frac{1}{2} z S(z)$ in (4.11): $\frac{1}{2} z S(z) = \rho + \mathrm{i} * \omega = \rho + \mathrm{i}\omega_{12}$. In particular

we conclude from the above that

$$\begin{aligned}\gamma &= \{x \in \Omega^+ : \omega_{1n}(x) = \cdots = \omega_{n-1,n}(x) = 0\}\\ &= \sigma^{-1}(\text{the } n\text{:th coordinate axis}).\end{aligned}$$

Let J_σ denote the Jacobian matrix of σ:

$$J_\sigma = \begin{pmatrix} \frac{\partial\omega_{1n}}{\partial x_1} & \frac{\partial\omega_{1n}}{\partial x_2} & \dots & \frac{\partial\omega_{1n}}{\partial x_n} \\ \frac{\partial\omega_{2n}}{\partial x_1} & \frac{\partial\omega_{2n}}{\partial x_2} & \dots & \frac{\partial\omega_{2n}}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial\omega_{n-1,n}}{\partial x_1} & \frac{\partial\omega_{n-1,n}}{\partial x_2} & \dots & \frac{\partial\omega_{n-1,n}}{\partial x_n} \\ \frac{\partial\rho}{\partial x_1} & \frac{\partial\rho}{\partial x_2} & \dots & \frac{\partial\rho}{\partial x_n} \end{pmatrix}.$$

Equation (4.8) with $k = n$ says that

$$\operatorname{tr} J_\sigma = 0.$$

Equivalently, if σ considered as a vector field,

$$\operatorname{div} \sigma = 0.$$

Note also, by Corollary 4.3, that σ is a harmonic map (each component of σ is a harmonic function).

Clearly σ extends continuously to $\overline{\Omega^+}$. On $(\partial\Omega)^+$,

$$\sigma = (0, \dots, 0, \frac{1}{2}r^2), \tag{4.14}$$

hence $(\partial\Omega)^+$, of dimension $n-1$, is mapped by σ onto a set of dimension at most one. It follows that $\operatorname{rank} J_\sigma \le 2$ at all points of $(\partial\Omega)^+$ (since σ loses at least $n-2$ dimensions). To be more precise, for any vector $\boldsymbol{\eta}$,

$$J_\sigma \boldsymbol{\eta} = \begin{pmatrix} \boldsymbol{\eta} \cdot \nabla\omega_{1n} \\ \boldsymbol{\eta} \cdot \nabla\omega_{2n} \\ \dots \\ \boldsymbol{\eta} \cdot \nabla\omega_{n-1,n} \\ \boldsymbol{\eta} \cdot \nabla\rho \end{pmatrix},$$

and writing, at a given point on $(\partial\Omega)^+$, $\boldsymbol{\eta}$ as

$$\boldsymbol{\eta} = \boldsymbol{\eta}_t + \eta_n\, \mathbf{n}$$

with $\boldsymbol{\eta}_t$ tangent to $(\partial\Omega)^+$ and $\eta_n \in \mathbb{R}$ gives (since $\omega_{jn} = 0$ on $(\partial\Omega)^+$),

$$J_\sigma \boldsymbol{\eta} = (\boldsymbol{\eta}_t \cdot \nabla\rho)\, \mathbf{e}_n + \eta_n\, J_\sigma\,(\mathbf{n}) = (\boldsymbol{\eta}_t \cdot \mathbf{x})\, \mathbf{e}_n + \eta_n\, J_\sigma\,(\mathbf{n}),$$

where $\mathbf{e}_n$ is the n:th unit vector (for the last member Proposition 4.4 was used). Thus the range of J_σ is contained in the (at most) two dimensional vector space spanned by $\mathbf{e}_n$ and $J_\sigma(\mathbf{n})$.

At points on $(\partial\Omega)^+$ where $\frac{1}{2}r^2$ is stationary, i.e., $\boldsymbol{\xi} = 0$, we have $\boldsymbol{\eta}_t \cdot \nabla\rho = 0$, hence rank $J_\sigma \leq 1$. Since the trace of J_σ is zero it follows that all eigenvalues of J_σ are zero. In two dimensions it even follows that $J_\sigma = 0$, because J_σ is symmetric when $n = 2$.

Returning to the mapping properties of σ, we see from (4.14) that σ maps $(\partial\Omega)^+$ onto the segment $\frac{1}{2}r_0^2 < \sigma_n < \frac{1}{2}r_1^2$ of the σ_n-axis, possibly except for endpoints, where

$$r_0 = \inf_{(\partial\Omega)^+} r, \qquad r_1 = \sup_{(\partial\Omega)^+} r.$$

On $D = \Omega \cap \mathbb{R}^{n-1}$ we have

$$\begin{aligned} \sigma &= (x_1 \frac{\partial u}{\partial x_n}, \dots, x_{n-1}\frac{\partial u}{\partial x_n}, \frac{1}{2}r^2 - r\frac{\partial u}{\partial r} - (n-2)u) \\ &= (-\frac{x_1}{2}f(x'), \dots, -\frac{x_{n-1}}{2}f(x'), \frac{1}{2}r^2 - r\frac{\partial u}{\partial r} - (n-2)u). \end{aligned} \tag{4.15}$$

Thus, since we have assumed that $f > 0$ on D, $\sigma(D)$ meets the σ_n-axis only if $0 \in \Omega$. The meeting point is $\sigma(0) = (0, \dots, 0, -(n-2)u(0))$, hence has $\sigma_n(0) \leq 0$.

In summary, σ maps the boundary $\partial(\Omega^+)$ of Ω^+ as follows: $0 \in \partial(\Omega^+)$ is mapped onto a point $(0, \dots, 0, \sigma_n)$ with $\sigma_n = -(n-2)u(0) \leq 0$, $\partial(\Omega^+) \cap (\mathbb{R}^n)^+ = (\partial\Omega)^+$ is mapped to points on the σ_n-axis, with $\sigma_n \geq \frac{1}{2}r_0^2 > 0$, and for the remaining points $x \in D \setminus \{0\}$ of $\partial(\Omega^+)$, $\sigma(x)$ does not meet the σ_n-axis. More precisely, x and $\sigma(x)$ are by (4.15) located on opposite sides of the σ_n-axis. It follows from all this that $\partial(\Omega^+)$ encloses the interval $[-(n-2)u(0), \frac{1}{2}r_0^2]$ on the σ_n-axis $(-1)^{n-1}$ times in the sense of degree theory. Thus

Theorem 4.8. *Assume $0 \in \Omega$ and that μ is of the form (4.13) with $f > 0$ on D. Then the map σ has mapping degree (Brouwer degree) $(-1)^{n-1}$ with respect to each point $(0, \dots, 0, t)$ with $-(n-2)u(0) < t < \frac{r_0^2}{2}$.*

A conclusion is that each value $(0, \dots, 0, t)$, $0 < t < \frac{r_0^2}{2}$, is attained at least once, but it can also be attained several times with signatures (signs of the Jacobi determinant) adding up to $(-1)^{n-1}$.

In the case of two dimensions a stronger conclusion is possible because the Jacobi determinant has constant sign: $\det J_\sigma \leq 0$. Switching the order between ω and ρ one gets the analytic function $\frac{1}{2}zS(z) = \rho + \mathrm{i}*\omega$ (which we still denote $\sigma(z)$), with nonnegative Jacobi determinant, and all arguments become more familiar, e.g., the mapping degree can be identified with the argument variation divided by 2π. We repeat everything in this case, with stronger assertions.

Corollary 4.9. *With assumptions as in the theorem we have, in the case $n = 2$, that if $0 \in \Omega$ then there is exactly one point on $(\partial\Omega)^+$ of shortest distance to the origin and γ is an analytic smooth arc from origin to that point. In particular, the*

equivalent conditions in Theorem 3.1 hold. In addition, γ is an integral curve of $\boldsymbol{\xi}$.

Proof. Let $r_0 > 0$ be the distance from the origin to $(\partial\Omega)^+$. We apply the argument principle to the analytic function $\sigma(z) = \frac{1}{2}zS(z) = \rho + \mathrm{i} * \omega$ in Ω^+. We have $\sigma(0) = 0$, $\operatorname{Im}\sigma(x) > 0$ for $x < 0$, $\operatorname{Im}\sigma(x) < 0$ for $x > 0$. For $z \in (\partial\Omega)^+$, $\sigma(z) = \frac{1}{2}|z|^2$, hence $\sigma((\partial\Omega)^+) \subset [\frac{1}{2}r_0^2, \infty)$.

It follows that as z runs through $\partial(\Omega^+)$, then $\sigma(z)$ encircles each point on the open interval $(0, \frac{1}{2}r_0^2) \subset \mathbb{R}$ exactly once. In addition, no other point on $\mathbb{R}$ is encircled, although some points lie on the image curve $\sigma(\partial(\Omega^+))$. But now γ is exactly the inverse image of $\mathbb{R}$ under σ, hence it follows that σ is univalent on γ. This means that $t \mapsto \sigma^{-1}(t)$, for $0 < t < \frac{1}{2}r_0^2$, is a bijective analytic parametrization of γ.

Since $(\partial\Omega)^+$ is known to be analytic without singular points in the present geometry [10], [12], [6] the Schwarz function $S(z)$, and hence $\sigma(z)$, extends to be analytic in a full neighborhood of $(\partial\Omega)^+$. In particular, σ is an open mapping in such a neighborhood.

Let $z_0 \in (\partial\Omega)^+$ be a point on distance r_0 from the origin. Then, with the above extended σ, $\sigma(z_0) = \frac{1}{2}r_0^2$, and any neighborhood U of z_0 in $(\mathbb{R}^2)^+$ is mapped onto a full neighborhood $\sigma(U)$ of $\frac{1}{2}r_0^2$. The latter neighborhood contains the interval $(\frac{1}{2}r_0^2 - \varepsilon, \frac{1}{2}r_0^2)$ on the real axis for some $\varepsilon > 0$, hence U contains the final part of γ. Since this is true for any neighborhood U of z_0 it follows that $\overline{\gamma} \cap (\partial\Omega)^+$ consists of just the point z_0. In particular, there is only one point on $(\partial\Omega)^+$ on distance r_0 from the origin.

The last statement of the corollary follows from the fact that $L_{\boldsymbol{\xi}}(*\omega) = 0$ (see (4.10)) in two dimensions, or simply from the analyticity of $\frac{1}{2}zS(z) = \rho + \mathrm{i} * \omega$: γ is a level line of $*\omega$ and $\boldsymbol{\xi} = \nabla\rho$ is perpendicular to level lines of ρ, which are orthogonal to those of $*\omega$. □

Now we return to $n \geq 2$ dimensions. We still expect that there is only one point on $(\partial\Omega)^+$ of shortest distance to the origin, that γ reaches $(\partial\Omega)^+$ only at that point, and that there are no other stationary points of $\frac{1}{2}r^2$. The following proposition gives some partial results in this direction.

Proposition 4.10. *The closure of γ in $\mathbb{R}^n$ intersects $(\partial\Omega)^+$ only at points of $(\partial\Omega)^+$ where $\boldsymbol{\xi} = 0$, i.e., where $\frac{1}{2}r^2$ is stationary. If there is a strict local maximum of $\frac{1}{2}r^2$ at some point $x \in (\partial\Omega)^+$, then x is in the closure of γ.*

Proof. It is easy to see that, at points in Ω^+ close to $(\partial\Omega)^+$, ∇u is essentially directed in the inward normal direction. At any point $x \in (\partial\Omega)^+$ where $\boldsymbol{\xi} \neq 0$, the vector $\mathbf{x}$ from the origin to x has a nonzero angle to the normal vector. It follows from (4.12) that ∇u cannot be parallel to $\mathbf{x}$ close to such points, hence that γ does not reach points of $(\partial\Omega)^+$ where $\boldsymbol{\xi} \neq 0$.

Next, let $x \in (\partial\Omega)^+$ be a point at which $\frac{1}{2}r^2$ attains a strict local maximum. Then for some neighborhood N of x

$$(\overline{\Omega^+} \setminus \{x\}) \cap N \subset B(0,r) \cap N$$

$(r = |x|)$. For a slightly smaller r, say $r' < r$, $S = (\Omega^+ \cap \partial B(0,r')) \cap N$ is a piece of a sphere cut off by $(\partial\Omega)^+$. In this piece $u > 0$, while on the boundary $u = 0$. It follows that there is a point on S where $u|_S$ attains a local maximum. But at such a point the gradient ∇u is perpendicular to S, which means that the point belongs to γ. Since r' was arbitrarily close to r, this means that we can produce points on γ arbitrarily close to x. $\square$

4.6. The Hessian

Let

$$H = (\frac{\partial^2 u}{\partial x_i \partial x_j})$$

be the Hessian of u. It is a symmetric matrix with $\operatorname{tr} H = \Delta u = 1$ in Ω^+. The rank of H is at least one (since the sum of eigenvalues is different from zero). On $(\partial\Omega)^+$ the rank is exactly one since $\nabla u = 0$ on $(\partial\Omega)^+$ and hence all derivatives of ∇u in tangential directions of $(\partial\Omega)^+$ are zero.

From $r\frac{\partial}{\partial r}\nabla u = (\mathbf{x} \cdot \nabla)\nabla u = H\mathbf{x}$, where the last product is matrix multiplication, it follows that the definition of $\boldsymbol{\xi}$ can be written in terms of H as

$$\boldsymbol{\xi} = \mathbf{x} - (n-1)\nabla u - H\mathbf{x}.$$

Thus, on $(\partial\Omega)^+$,

$$H\mathbf{x} = \mathbf{x} - \boldsymbol{\xi},$$

and in particular

$$H\mathbf{x} = \mathbf{x}$$

at stationary points (for $\frac{1}{2}r^2$) on $(\partial\Omega)^+$.

More generally we can write, for any $\boldsymbol{\eta} \in \mathbb{R}^n$,

$$H\boldsymbol{\eta} = \nabla(\boldsymbol{\eta} \cdot \nabla u).$$

On $(\partial\Omega)^+$, $\boldsymbol{\eta} \cdot \nabla u = 0$, hence $\nabla(\boldsymbol{\eta} \cdot \nabla u)$ is perpendicular to $(\partial\Omega)^+$. Therefore, on $(\partial\Omega)^+$,

$$H\boldsymbol{\eta} = \lambda\mathbf{n}$$

for some $\lambda = \lambda(\boldsymbol{\eta}) \in \mathbb{R}$. It follows that $\mathbf{n}$ defines the only eigendirection for H corresponding to a nonzero eigenvalue, and since the sum of eigenvalues equals one, that

$$H\mathbf{n} = \mathbf{n}.$$

Moreover, it follows that H annihilates every vector tangent to $(\partial\Omega)^+$. Thus, on $(\partial\Omega)^+$, H simply is the orthogonal projection onto the normal of $(\partial\Omega)^+$:

$$H\boldsymbol{\eta} = (\boldsymbol{\eta} \cdot \mathbf{n})\mathbf{n}.$$

Proposition 4.11. *With $dx = dx_1 \dots dx_n = dm$,*

$$\int_{\Omega^+} \frac{\partial^2 u}{\partial x_i \partial x_j} dx = 0 \quad \big((i,j) \neq (n,n)\big),$$
$$\int_{\Omega^+} \frac{\partial^2 u}{\partial x_n^2} dx = \frac{1}{2} m(\Omega) = \frac{1}{2} \int d\mu.$$

Proof. For any entry $(i,j) \neq (n,n)$, say $i = 1$, j arbitrary, we have

$$\begin{aligned} \int_{\Omega^+} \frac{\partial^2 u}{\partial x_1 \partial x_j} dx_1 \dots dx_n &= \int_{\Omega^+} d\Big(\frac{\partial u}{\partial x_j} dx_2 \dots dx_n\Big) \\ &= \int_{\partial(\Omega^+)} \frac{\partial u}{\partial x_j} dx_2 \dots dx_n = 0 \end{aligned}$$

(note that $dx_n = 0$ on $\mathbb{R}^{n-1}$), while for $i = j = n$, using the above

$$\int_{\Omega^+} \frac{\partial^2 u}{\partial x_n^2} dx = \int_{\Omega^+} \Delta u dx = \frac{1}{2} m(\Omega) = \frac{1}{2} \int d\mu.$$

□

4.7. Convexity questions

We finish by some simple observations related to convexity.

Proposition 4.12. *The restriction of U^μ to any hemisphere $(\partial B(a,R))^+$ with center a on $\mathbb{R}^{n-1}$ is convex as a function of $x' = (x_1, \dots, x_{n-1})$.*

Proof. This follows from a straight-forward computation of the second derivatives of $x' \mapsto U^\mu(x', \sqrt{R^2 - |x' - a|^2})$. Assuming for simplicity that $a = 0$ and computing, e.g., the derivatives in the x_1-direction in the case $n \geq 3$ we have, with $x_n^2 = R^2 - |x'|^2$,

$$\begin{aligned} U^\mu(x', \sqrt{R^2 - |x'|^2}) = U^\mu(x', x_n) &= \frac{1}{(n-2)|S^{n-1}|} \int \frac{d\mu(y')}{(|x' - y'|^2 + x_n^2)^{\frac{n-2}{2}}} \\ &= \frac{1}{(n-2)|S^{n-1}|} \int \frac{d\mu(y')}{(|y'|^2 - 2x' \cdot y' + R^2)^{\frac{n-2}{2}}}, \\ \frac{\partial U^\mu}{\partial x_1} &= \frac{1}{(n-2)|S^{n-1}|} \int \frac{(n-2) y_1 \, d\mu(y')}{(|y'|^2 - 2x' \cdot y' + R^2)^{\frac{n}{2}}}, \\ \frac{\partial^2 U^\mu}{\partial x_1^2} &= \frac{1}{(n-2)|S^{n-1}|} \int \frac{n(n-2) y_1^2 \, d\mu(y')}{(|y'|^2 - 2x' \cdot y' + R^2)^{\frac{n+2}{2}}} \geq 0, \end{aligned}$$

where $|S^{n-1}|$ denotes the spherical area of S^{n-1}.

□

Proposition 4.13. *Assume that $\operatorname{supp} \mu \subset B(a,R) \cap \mathbb{R}^{n-1}$ for a ball $B(a,R)$ with center $a \in \mathbb{R}^{n-1}$. Then the density of classical balayage of μ onto the sphere $\partial B(a,R)$ is convex as a function of x'.*

Proof. We may assume that $a = 0$. Set $B_R = B(0, R)$. Classical balayage of a point mass at any point $y \in B_R$ onto ∂B_R gives the mass distribution on ∂B_R with density equal to the Poisson kernel (or normal derivative of Green function), i.e., the function of $x \in \partial B_R$

$$P_{B_R}(x, y) = \frac{1}{|S^{n-1}|R} \frac{R^2 - |y|^2}{|x - y|^n} = \frac{1}{|S^{n-1}|R} \frac{R^2 - |y|^2}{(|y|^2 - 2x \cdot y + R^2)^{\frac{n}{2}}}.$$

It follows that the density β_0 of classical balayage of μ is obtained by integrating the above with respect to $y' \in \mathbb{R}^{n-1}$:

$$\begin{aligned} \beta_0(x, B_R) &= \frac{1}{|S^{n-1}|R} \int \frac{R^2 - |y'|^2}{(|y'|^2 - 2x \cdot y' + R^2)^{\frac{n}{2}}} d\mu(y') \\ &= \frac{1}{|S^{n-1}|R} \int \frac{R^2 - |y'|^2}{(|y'|^2 - 2x' \cdot y' + R^2)^{\frac{n}{2}}} d\mu(y'). \end{aligned}$$

Note that the last expression is a function only of x'. One sees immediately that as such a function the second order derivative in any direction is nonnegative (compare previous proof). In fact, any derivative of even order is nonnegative (a similar remark applies to the previous proposition).

$\square$

References

[1] L. A. Caffarelli, *The obstacle problem revisited*, J. Fourier Anal. Appl. **4** (1998), 383–402.

[2] P. J. Davis: *The Schwarz Function and its Applications*, Carus Mathematical Monographs, Math. Assoc. Amer., 1974.

[3] T. Frankel: *The Geometry of Physics. An Introduction.* Cambridge Univ. Press, Cambridge, UK, 1997.

[4] A. Friedman: *Variational Principles and Free Boundaries*, Wiley and Sons, 1982.

[5] B. Gustafsson: *On quadrature domains and an inverse problem in potential theory*, J. Analyse Math. **55** (1990), 172–216.

[6] B. Gustafsson, M. Sakai, *Properties of some balayage operators with applications to quadrature domains and moving boundary problems*, Nonlinear Anal. **22** (1994), 1221–1245.

[7] B. Gustafsson, M. Sakai, *Sharp estimates of the curvature of some free boundaries in two dimensions*, Ann. Acad. Sci. Fenn. Math. **28** (2003), 123–142.

[8] B. Gustafsson, M. Sakai, *On the curvature of the free boundary for the obstacle problem in two dimensions*, Monatshefte für Mathematik **142** (2004), 1–5.

[9] L. Hörmander: *Notions of Convexity*, Birkhäuser, Boston, 1994.

[10] M. Sakai, *Quadrature Domains*, Lect. Notes Math. **934**, Springer-Verlag, Berlin-Heidelberg 1982.

[11] M. Sakai, *Application of variational inequalities to the existence theorem on quadrature domains*, Trans. Amer. Math. Soc. **276** (1983), 267–279.

[12] M. Sakai, *Regularity of boundary having a Schwarz function*, Acta Math. **166** (1991), 263–297.

[13] M. Sakai, *Small modifications of quadrature domains*, Memoirs of the American mathematical Society, **969**, ISSN 0065-9266, Providence, Rhode Island, 2010.

[14] D. G. Schaeffer, *One-sided estimates for the curvature of the free boundary in the obstacle problem*, Adv. Math. **24** (1977), 78–98.

[15] H. S. Shapiro, *The Schwarz Function and its Generalizations to Higher Dimensions*, Univ. of Arkansas Lect. Notes Math. Vol. 9, Wiley, New York, 1992.

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